RIKEN-TH-96

Supersymmetric Field Theory Based on Generalized Uncertainty Principle

SHIBUSA Yuuichirou

Theoretical Physics Laboratory,

RIKEN (The Institute of Physical and Chemical Research),

Wako, 351-0198, Japan

Abstract

We construct a quantum theory of free fermion field based on the deformed Heisenberg algebra $[\hat{x}, \hat{p}] = i\hbar(1+\beta\hat{p}^2)$ where β is a deformation parameter using supersymmetry as a guiding principle. A supersymmetric field theory with a real scalar field and a Majorana fermion field is given explicitly and we also find that the supersymmetry algebra is deformed from an usual one.

PACS numbers: 03.65.Ca, 03.70.+k, 11.10.Ef, 11.30.Pb

I. INTRODUCTION

Physics in extremely high energy regions is particularly of interest to particle physics. In particular, when we discuss gravity, it is expected that there is a minimal length in principle. String theory which has a characteristic scale $\sqrt{\alpha'}$, is one of the most successful theoretical frameworks which overcome the difficulty of ultra-violet divergence in quantum theory of gravity. However, string theory has many difficulties in performing practical computations. Therefore if we construct a field theory which captures some stringy nature and/or includes stringy corrections, it would play a pivotal role in investigating physics in high energy regions even near the Planck scale.

Some of the stringy corrections appear as α' corrections. In other words, it often takes the form as higher derivative corrections i.e. higher order polynomial of momentum. One way to discuss these corrections is deforming the Heisenberg uncertainty principle to a generalized uncertainty principle (GUP):

$$\Delta\hat{x} \geq \frac{\hbar}{2\Delta\hat{p}} + \frac{\hbar\beta}{2}\Delta\hat{p}, \tag{1.1}$$

where β is a deforming parameter and corresponds to the square of the minimal length scale. If GUP is realized in a certain string theory context, β would take a value of order the string scale ($\beta \sim \alpha'$). This relation comes from various types of studies such as on high energy or short distance behavior of strings [1], [2], gedanken experiment of black hole [3], de Sitter space [4], the symmetry of massless particle [5] and wave packets [6].

There are several canonical commutation algebra which lead to the GUP. Among these algebra we will focus on the algebra;

$$[\hat{x}, \hat{p}] = i\hbar(1 + \beta\hat{p}^2). \tag{1.2}$$

This algebra is investigated in [7]-[10] and an attempt to construct a field theory with minimal length scale is made in [11] by using the Bargmann-Fock representation in 1+1 dimensional spacetime. It has also been used in cosmology, especially in physics at an early universe (see for example, [12]-[15] and references therein).

In our previous paper [16], we investigated the quantization of fields based on the deformed algebra (1.2) in the canonical formalism in 1+1 dimensions and in the path integral formalism as well. Using the path integral formalism we constructed a quantum theory of

scalar field in arbitrary spacetime dimensions. This theory has a non-locality which stems from the existence of a minimal length.

In this paper, we construct a quantum theory of free fermion field based on the deformed Heisenberg algebra. Where, we respect supersymmetry as a guiding principle. This is because a string theory has this symmetry and we intend to construct a field theory which contains the stringy corrections. Moreover, supersymmetry is also an useful tool to understand physics in ultra-violet momentum regions. It manages a behavior of system in extremely high energy regions and eases ultra-violet divergence in quantum theory. Therefore we propose a quantum field theory of fermion to have a supersymmetry for a scalar system which was given in [16]. In two and three-dimensional spacetime, we give a system with one real scalar and one Majorana fermion explicitly. This system has a special symmetry between a boson and a fermion which corresponds to supersymmetry. Although, this symmetry is deformed from ordinary supersymmetry. From the fermionic part of this system, we propose an action of fermionic fields based on GUP in general dimensional spacetime.

II. SCALAR FIELD THEORY

In the paper [16], we proposed a field theory of scalar based on GUP in the path-integral formalism. We begin with a review of this theory.

Our theory is based on the following algebra [7]:

$$\left[\hat{x}^i, \hat{p}_j\right] \;=\; i\hbar(1+\beta\hat{\boldsymbol{p}}^2)\delta^i_j. \tag{2.1}$$

This is an extension to higher dimensional spacetime of deformed Heisenberg algebra (1.2). Here i, j run from 1 to d which is the number of spatial coordinates and $\hat{\boldsymbol{p}}^2 \equiv \sum_{i=1}^{d}(\hat{p}_i)^2$. Hereinafter, we use index i, j for spatial coordinates and a, b for all spacetime coordinates. Jacobi identity determines the full algebra:

$$\left[\hat{x}^i, \hat{x}^j\right] \;=\; -2i\hbar\beta(1+\beta\hat{\boldsymbol{p}}^2)\hat{L}^{ij}. \tag{2.2}$$

$$\left[\hat{p}^i, \hat{p}^j\right] \;=\; 0. \tag{2.3}$$

Here $\hat{L}^{ij}$ are angular momentum like operators $\hat{L}^{ij} \equiv \frac{1}{2(1+\beta\hat{\boldsymbol{p}}^2)}(\hat{x}^i\hat{p}^j - \hat{x}^j\hat{p}^i + \hat{p}^j\hat{x}^i - \hat{p}^i\hat{x}^j)$. Because operators $\hat{p}^i$ commute with each other, we construct a theory in momentum space representation. In momentum space representation, momentum operators are diagonalized

simultaneously and we do not distinguish eigenvalues of momentum p_i from operators $\hat{p}_i$. In the following, we set Planck constant $\hbar$ to be 1 for simplicity.

Lagrangian in $d+1$ dimensional spacetime [16] is

$$\mathcal{L} = -\frac{1}{2}\int_{-\infty}^{\infty} d^d p(1+\beta\boldsymbol{p}^2)^{-1}\phi(-p,t)\left[\partial_t^2+\boldsymbol{p}^2+m^2\right]\phi(p,t), \tag{2.4}$$

$$\text{where, } \boldsymbol{p}^2 \equiv \sum_{i=1}^{d}(p_i)^2.$$

The difference from ordinary quantum field theory is a prefactor $(1+\beta\boldsymbol{p}^2)^{-1}$ in Lagrangian. Using the Bjorken-Johnson-Low prescription[17], from behavior of T*-product between $\phi(p,t)$ and $\phi(p',t')$, we obtain canonical commutation relation:

$$[\phi(p,t),\partial_t\phi(p',t)] = i(1+\beta\boldsymbol{p}^2)\delta^d(p+p'). \tag{2.5}$$

As we can see from this equation, a deforming prefactor $(1+\beta\hat{\boldsymbol{p}}^2)$ of Heisenberg algebra in the first quantization (2.1) also appears in canonical commutation relation of the second quantized field theory.

In a fermion field case, we encounter a difficulty at constructing the second quantized Hilbert space which does not appear in a scalar system. Note that a system of spin 0 particles contains only spin 0 particle. By contrast, a system of spin $\frac{1}{2}$ particles is not closed with only fermions in the sense that it contains bosons as bound states. Therefore algebra of fermion fields must be introduced to be consistent with that of bosons fields. Because the scalar fields in our theory have a different commutation relation (2.5) from ordinary one, we must construct fermion fields so that the composite fields which correspond to scalar particles have the same commutation relations. Or, in two-dimensional ordinary quantum field theory we could use the concepts of bosonization and fermionization which associate fermion fields with boson fields. However, it is obscure which of these principles which relate bosons and fermions remains unchanged in GUP or in extremely high energy regions. Instead of handling this problem directly, we use supersymmetry to construct quantized field theory of fermion. This is because string theory accommodates this symmetry and therefore it is expected that this symmetry is reflected in GUP or in extremely high energy regions.

In the next section we construct a quantum field theory of fermions which is consistent with the above scalar theory by using supersymmetry.

III. SUPERSYMMETRY IN GUP

In two and three-dimensional spacetime, a system with a real scalar and a Majorana fermion has a special symmetry between a boson and a fermion, namely supersymmetry. Thus we construct a quantum field theory of fermion in GUP to have a similar symmetry between bosons and fermions with an above-reviewed scalar system in two and three-dimensional spacetime.

Our notation for two and three-dimensional spacetime is as follows: In those dimensional spacetime (with signature $-+$ or $-++$) the Lorentz group has a real (Majorana) two-component spinor representation ψ^α. In the following, we explain the notation of three-dimensional spacetime. Reduction to two-dimensional spacetime is trivial. We define a representation of Gamma matrices by Pauli matrices[1] as follows:

$$\{\Gamma^a, \Gamma^b\} = 2\eta^{ab} = 2\mathrm{diag}(-++), \tag{3.1}$$

$$\Gamma^0 = -i\sigma_2, \Gamma^1 = \sigma_1, \Gamma^2 = -\sigma_3. \tag{3.2}$$

Spinor indices are lowered and raised by charge conjugation matrix $C_{\alpha\beta} \equiv \Gamma^0$ and its inverse matrix C^{-1}:

$$\psi_\alpha = \psi^\beta C_{\beta\alpha}(=\bar{\psi}_\alpha), \psi^\alpha = \psi_\beta (C^{-1})^{\beta\alpha}. \tag{3.3}$$

Because the algebra of scalar field (2.5) is deformed from usual one, it is natural to expect that supersymmetry algebra may also be deformed from ordinary one. We generalize supersymmetry algebra and its actions on a scalar field ϕ, a Majorana fermion ψ and an auxiliary field F with parameter ϵ^αas follows:

$$\left[\bar{\epsilon}_1\hat{Q}, \bar{\epsilon}_2\hat{Q}\right] = 2\Delta\bar{\epsilon}_1\Gamma^a\epsilon_2\hat{P}_a, \tag{3.4}$$

$$\delta\phi(p,t) = i\bar{\epsilon}\psi(p,t), \tag{3.5}$$

$$\delta\psi^\alpha(p,t) = A_1 F(p,t)\epsilon^\alpha - A_2\{(\bar{\epsilon}\Gamma^0 C^{-1})^\alpha\partial_t + (\bar{\epsilon}\Gamma^j C^{-1})^\alpha(ip_j)\}\phi(p,t), \tag{3.6}$$

$$\delta F(p,t) = A_3 i\bar{\epsilon}(\Gamma^0\partial_t + \Gamma^j(ip_j))\psi(p,t). \tag{3.7}$$

Here, we introduce factors Δ, A_i as functions of a deforming parameter β and momentum. These factors should reduce to 1 in the limit of $\beta \to 0$ and will be determined later by consistency conditions.

---

[1] Pauli matrices are $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

From the closeness of algebra on each fields, we obtain conditions

$$A_1 A_3 = A_2 = \Delta. \tag{3.8}$$

We also generalize a Lagrangian by introducing factors B_i, which are functions of a deforming parameter β and momentum and are to be determined as well:

$$\mathcal{L} = \int dp^d \quad \left\{ -\frac{B_1}{2}\phi(-p,t)(\partial_t^2 + \boldsymbol{p}^2)\phi(p,t) - \frac{iB_2}{2}\bar{\psi}(-p,t)(\Gamma^0\partial_t + (ip_i)\Gamma^i + m)\psi(p,t) \right.$$
$$\left. + \sqrt{B_1 B_3} m\phi(-p,t)F(p,t) + \frac{B_3}{2}F(-p,t)F(p,t) \right\}. \tag{3.9}$$

Here d is the number of spatial coordinates (1 or 2). By integrating out the field F, we obtain Lagrangian with the scalar field and the Majorana field:

$$\mathcal{L} = \int dp^d \quad \left\{ \frac{B_1}{2}\phi(-p,t)(\partial_t^2 + \boldsymbol{p}^2 + m^2)\phi(p,t) \right.$$
$$\left. - \frac{iB_2}{2}\bar{\psi}(-p,t)(\Gamma^0\partial_t + (ip_i)\Gamma^i + m)\psi(p,t) \right\}. \tag{3.10}$$

Invariance of Lagrangian (3.9) under supersymmetry variations (3.5)-(3.7) leads following conditions;

$$\begin{aligned} A_1 B_2 &= \sqrt{B_1 B_3}, \\ A_1 B_2 &= A_3 B_3, \\ B_1 &= A_1 A_3 B_2. \end{aligned} \tag{3.11}$$

From conditions (3.8) and (3.11), only B_1 and B_2 remain to be determined. (Factor A_1 can be absorbed into normalization of a field F and we set it to be 1 for a field F to be an auxiliary field.) Noether's current for supersymmetry can be calculated from Lagrangian (3.10) and supersymmetry charge is found to be

$$Q^\alpha = \int dt dp^d B_1 \{ \; - \; \psi^\alpha(-p,t)\partial_t\phi(p,t) + (\Gamma^i\Gamma^0\psi(-p,t))^\alpha(ip_i)\phi(p,t)$$
$$+ \; m(\Gamma^0\psi(-p,t))^\alpha\phi(p,t)\}. \tag{3.12}$$

Then, we obtain Hamiltonian of this system from supersymmetry charge and algebra (3.4),

$$\mathcal{H} \; = \; P^0 = -\frac{1}{4}\frac{B_2}{B_1}(C\Gamma^0)_{\alpha\beta}\{Q^\alpha, Q^\beta\}. \tag{3.13}$$

Using the Bjorken-Johnson-Low prescription, from behaviors of T*-product between fields, we obtain canonical commutation relations as follows,

$$[\phi(p,t),\partial_t\phi(q,t)] \;=\; \frac{i}{B_1}\delta(p+q), \tag{3.14}$$

$$\left\{\psi^\alpha(p,t),\psi^\beta(q,t)\right\} \;=\; -\frac{(\Gamma^0 C^{-1})^{\alpha\beta}}{B_2}\delta(p+q). \tag{3.15}$$

Thus we can write the Hamiltonian in the following form;

$$\begin{aligned}\mathcal{H} = \int dp^d \quad & \frac{B_1}{2}\{\pi(-p,t)\pi(p,t)+\phi(-p,t)(\boldsymbol{p}^2+m^2)\phi(p,t)\} \\ & +\frac{iB_2}{2}\bar{\psi}(-p,t)((ip_i)\Gamma^i+m)\psi(p,t).\end{aligned} \tag{3.16}$$

Here, we use conjugate momentum $\pi(p,t)=\partial_t\phi(-p,t)$ and indices i runs from 1 to d.

There is another condition which can be used to determine the factors B_1 and B_2. It comes from the free energy of supersymmetric vacuum. From algebra (3.4), supersymmetric state has zero energy:

$$0 \;=\; \frac{1}{2}\mathrm{Tr_B}\ln(B_1(E^2+\boldsymbol{p}^2+m^2)) - \frac{1}{4}\mathrm{Tr_F}\ln(B_2^2(E^2+\boldsymbol{p}^2+m^2)). \tag{3.17}$$

This fact leads to the condition;

$$B_1 \;=\; B_2^2. \tag{3.18}$$

Here $\mathrm{Tr_B}$ and $\mathrm{Tr_F}$ represent trace in bosonic and fermionic Hilbert space respectively.

Lastly, we set $B_1=(1+\beta\boldsymbol{p}^2)^{-1}$ as we can see from the scalar action (2.4). This determines all of the introduced factors as follows;

$$\Delta = A_2 = A_3 = B_2 \;=\; (1+\beta\boldsymbol{p}^2)^{-\frac{1}{2}}, \tag{3.19}$$

$$A_1 = B_3 \;=\; 1, \tag{3.20}$$

$$B_1 \;=\; (1+\beta\boldsymbol{p}^2)^{-1}. \tag{3.21}$$

Thus we construct quantized fields of fermion which is consistent with scalar fields (2.5) as

$$\left\{\psi^\alpha(p,t),\psi^\beta(q,t)\right\} \;=\; -(1+\beta\boldsymbol{p}^2)^{\frac{1}{2}}(\Gamma^0 C^{-1})^{\alpha\beta}\delta(p+q). \tag{3.22}$$

Note that a factor Δ is not equal to 1 no matter how we set A_1. Therefore this supersymmetry algebra is deformed from an usual one as

$$[\bar{\epsilon}_1 Q,\bar{\epsilon}_2 Q] \;=\; 2(1+\beta\boldsymbol{p}^2)^{-\frac{1}{2}}\bar{\epsilon}_1\Gamma^a\epsilon_2 P_a. \tag{3.23}$$

There is no difficulty in generalizing the above quantum fields of fermion to higher $d+1$ dimensions than three dimensions. The action is as follows:

$$\mathcal{L} = \int dp^d \quad \left\{ -\frac{i}{(1+\beta\boldsymbol{p}^2)^{\frac{1}{2}}} \bar{\psi}(-p,t)(\Gamma^0\partial_t + (ip_i)\Gamma^i + m)\psi(p,t) \right\}. \tag{3.24}$$

There appears a universal prefactor $(1+\beta\boldsymbol{p}^2)^{-\frac{1}{2}}$ comparing with usual fermion action regardless as to whether there were supersymmetry or not. This prefactor ensures that fermion fields are compatible with the scalar fields which had been constructed in our previous paper [16].

From the actions (2.4) and (3.24), we also have supersymmetric field theory in four dimensions with an complex scalar and a Majorana (or Weyl) fermion just as a corresponding ordinary field theory has supersymmetry in four dimensions.

IV. CONCLUSION AND DISCUSSIONS

In summary, we have constructed a quantum theory of free fermion field based on the deformed Heisenberg algebra. It is consistent with already proposed scalar theory through supersymmetry. We start with a system with an real scalar and a Majorana fermion in two- and three-dimensional spacetime and determine supersymmetric action. We found that supersymmetry algebra is deformed from an usual one. An extension to higher dimensions are trivial and there is also supersymmetric theory in four-dimensional spacetime.

We conclude with a brief discussion on Lorentz invariant extension of our theory. Lorentz invariant extension of deformed Heisenberg algebra (2.1) is known as a sort of 'doubly special relativity' or 'κ-deformation' (for example, see [18], [19] and references therein),

$$[\hat{x}^a, \hat{p}_b] \; = \; i\hbar(1+\beta\hat{p}^2)\delta^a_b. \tag{4.1}$$

Here a, b run from 0 to d and $\hat{p}^2 \equiv -(\hat{p}^0)^2 + \sum_{i=0}^{d}(\hat{p}_i)^2$. Thus we claim that an action where the factor $(1+\beta\boldsymbol{p}^2)$ is replaced with a new factor $(1+\beta p^2)$ describes quantum field theory of doubly special relativity. In such case, time slice is not well-defined because of the existence of minimal time interval. Therefore there is no canonical formalism.

Acknowledgments

The author is grateful to T. Matsuo, K. Oda, T. Tada, and N. Yokoi for valuable discussions. The author is supported by the Special Postdoctoral Researchers Program at RIKEN.

---

[1] D. J. Gross and P. F. Mende, Phys. Lett. B **197** (1987) 129.

[2] K. Konishi, G. Paffuti and P. Provero, Phys. Lett. B **234** (1990) 276.

[3] M. Maggiore, Phys. Lett. B **304** (1993) 65 [arXiv:hep-th/9301067].

[4] H. S. Snyder, Phys. Rev. **71** (1947) 38.

[5] W. Chagas-Filho, arXiv:hep-th/0505183.

[6] J. Y. Bang and M. S. Berger, Phys. Rev. D **74** (2006) 125012 [arXiv:gr-qc/0610056].

[7] A. Kempf, G. Mangano and R. B. Mann, Phys. Rev. D **52** (1995) 1108 [arXiv:hep-th/9412167].

[8] S. Hossenfelder, M. Bleicher, S. Hofmann, J. Ruppert, S. Scherer and H. Stoecker, Phys. Lett. B **575** (2003) 85 [arXiv:hep-th/0305262].

[9] S. Hossenfelder, Phys. Rev. D **70** (2004) 105003 [arXiv:hep-ph/0405127].

[10] S. Hossenfelder, arXiv:hep-th/0510245.

[11] A. Kempf, J. Math. Phys. **38** (1997) 1347 [arXiv:hep-th/9602085].

[12] A. Ashoorioon, A. Kempf and R. B. Mann, Phys. Rev. D **71** (2005) 023503 [arXiv:astro-ph/0410139].

[13] A. Ashoorioon and R. B. Mann, Phys. Rev. D **71** (2005) 103509 [arXiv:gr-qc/0410053].

[14] A. Ashoorioon and R. B. Mann, Nucl. Phys. B **716** (2005) 261 [arXiv:gr-qc/0411056].

[15] A. Ashoorioon, J. L. Hovdebo and R. B. Mann, Nucl. Phys. B **727** (2005) 63 [arXiv:gr-qc/0504135].

[16] T. Matsuo and Y. Shibusa, Mod. Phys. Lett. A **21** (2006) 1285 [arXiv:hep-th/0511031].

[17] J. D. Bjorken, Phys. Rev. **148** (1966) 1467.
K. Johnson and F. E. Low, Prog. Theor. Phys. Suppl. **37** (1966) 74.

[18] J. Lukierski and A. Nowicki, Int. J. Mod. Phys. A **18** (2003) 7 [arXiv:hep-th/0203065].

[19] G. Amelino-Camelia, Int. J. Mod. Phys. D **11** (2002) 1643 [arXiv:gr-qc/0210063].